Mechanical Model for Relativistic Blast Waves

Andrei M. Beloborodov,[1] Zuhngwhi Lucas Uhm

Physics Department and Columbia Astrophysics Laboratory,
Columbia University, 538 West 120th Street, New York, NY 10027.

ABSTRACT

Relativistic blast waves can be described by a mechanical model. In this model, the "blast" — the compressed gas between the forward and reverse shocks — is viewed as one hot body. Equations governing its dynamics are derived from conservation of mass, energy, and momentum. Simple analytical solutions are obtained in the two limiting cases of ultra-relativistic and non-relativistic reverse shock. Equations are derived for the general explosion problem.

Subject headings: gamma rays: bursts—hydrodynamics—relativity—shock waves

1. Introduction

Relativistic blast waves are believed to produce afterglow emission of gamma-ray bursts (GRBs). The blast wave is driven by a shell with Lorentz factor $\Gamma_{\rm ej} \sim 10^2 - 10^3$ which is ejected by a central trigger of the explosion. The ejecta expands and drives a forward shock (FS) into the external medium, and a reverse shock (RS) propagates inside the ejecta. This standard explosion picture has four regions: 1 — external medium, 2 — shocked external medium, 3 — shocked ejecta, and 4 — unshocked ejecta (e.g. Piran 2004).

The full hydrodynamical simulations of ultra-relativistic blast waves are expensive, while GRB modeling requires exploration of a broad range of models. Therefore, approximate hydrodynamical calculations are commonly used. A customary approximation divided the explosion into two stages: before and after the RS crosses the ejecta. At first stage, pressure balance across the blast wave was assumed, i.e. the pressures at FS and RS were equated, $p_f = p_r$. At the second stage, the self-similar solution of Blandford & McKee (1976) was applied. This description, however, has two drawbacks: (1) the approximation $p_f = p_r$ violates energy conservation for adiabatic blast waves (by as much as a factor of 3 in some

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[1]Also at Astro-Space Center, Lebedev Physical Institute, Profsojuznaja 84/32, Moscow 117810, Russia

example explosions that we studied), and (2) the model has to assume a sharp trailing edge of the ejecta, so that p_r suddenly drops to zero when RS reaches the edge. A realistic model should describe the transition to regime $p_r \ll p_f$; the transition is especially interesting if the ejecta has a long tail.

In this paper, we propose a "mechanical" model which does not have these problems. Like the previously used approximation, we assume that regions 2 and 3 move with a common Lorentz factor Γ. Thus, in essence we replace the gas between FS and RS by one body, which we call "blast." This is reasonable since internal motions in the blast are subsonic, and hydrodynamical simulations confirm that $\Gamma \approx const$ between the FS and RS (Kobayashi & Sari 2000). Instead of finding Γ from the condition $p_f = p_r$, we leave Γ as a dynamical variable, and find its evolution from a differential equation expressing momentum conservation. For a given instantaneous Γ, pressures p_f and p_r are found from the jump conditions at the two shock fronts, and the difference $(p_f - p_r)$ governs the evolution of Γ.

Our model assumes spherical symmetry. However, it remains valid if the explosion is driven by a jet with a small opening angle $\theta_{\rm jet}$ as long as $\Gamma \gg \theta_{\rm jet}^{-1}$. Then the jet behaves like a portion of spherical ejecta since the edge of the jet is causally disconnected from its axis.

In § 2, we present a useful form of the jump conditions for a shock with arbitrary Lorentz factor and apply them to the FS and RS of a relativistic blast wave. The equations of mechanical model and analytical solutions are derived in § 3.

2. JUMP CONDITIONS

Ideal gas has stress-energy tensor $T^{\alpha\beta} = (e+p)u^\alpha u^\beta + g^{\alpha\beta}p$ and mass flux $j^\alpha = \rho u^\alpha$, where $g^{\alpha\beta}$ is Minkowski metric, u^α is 4-velocity of the gas, ρ is rest-mass density, e is energy density (including rest energy), and p is pressure; all the thermodynamic quantities are defined in the rest frame of the gas. A shock is described by three jump conditions that express continuity of j^α and $T^{\alpha\beta}$ in the shock frame (Landau & Lifshitz 1959),

$$\gamma_2\beta_2\,\rho_2 = \gamma_1\beta_1\,\rho_1, \qquad \gamma_2^2\beta_2\,(e_2+p_2) = \gamma_1^2\beta_1\,\rho_1 c^2, \qquad \gamma_2^2\beta_2^2\,(e_2+p_2) + p_2 = \gamma_1^2\beta_1^2\,\rho_1 c^2.$$

Here subscripts 1 and 2 refer to preshock (cold) and postshock (hot) medium, β_1 and β_2 are the gas velocities relative to the shock front, and we assumed $p_1 = 0$ and $e_1 = \rho_1 c^2$. The shock strength may be described by relative velocity $\beta_{12} = (\beta_1 - \beta_2)/(1-\beta_1\beta_2)$ or $\gamma_{12} = (1-\beta_{12}^2)^{-1/2}$. The postshock gas satisfies $e_2/\rho_2 c^2 = \gamma_{12}$, and a convenient approximation for p_2 is

$$p_2 = \frac{1}{3}\left(1 + \frac{1}{\gamma_{12}}\right)(e_2 - \rho_2 c^2). \qquad (1)$$

It is exact if the gas is monoenergetic, i.e., particles have equal energies in the gas frame. Its error for Maxwellian gas is within 5%. Using equation (1), we express all quantities in terms of β_{12} (or γ_{12}), which remains as the only free parameter of the shock,

$$\beta_2 = \frac{\beta_{12}}{3}, \qquad \rho_2 = 4\gamma_{12}\rho_1, \qquad e_2 = 4\gamma_{12}^2\,\rho_1 c^2, \qquad p_2 = \frac{4}{3}\,(\gamma_{12}^2 - 1)\,\rho_1 c^2. \tag{2}$$

These equations apply to shocks of arbitrary strength, relativistic or non-relativistic.

The blast wave has two shocks, forward and reverse. The above equations with $\gamma_{12} = \Gamma$ describe FS. The RS is described by the same equations when index 1 is replaced by 4 and index 2 is replaced by 3. Pressures $p_f = p_2$ and $p_r = p_3$ are given by

$$p_f = \frac{4}{3}\,(\Gamma^2 - 1)\,\rho_1 c^2, \qquad p_r = \frac{4}{3}\,(\gamma_{43}^2 - 1)\,\rho_4 c^2. \tag{3}$$

The approximation of common Lorentz factor of regions 2 and 3 implies that Γ and γ_{43} are not independent. We assume that the Lorentz factor of unshocked ejecta, $\Gamma_{\rm ej}$, is known in the lab frame. Then one finds its Lorentz factor relative to the blast, $\gamma_{43} = \Gamma\Gamma_{\rm ej}(1 - \beta\beta_{\rm ej})$. For relativistic blast waves ($\Gamma_{\rm ej} \gg 1$ and $\Gamma \gg 1$) this relation simplifies,

$$\gamma_{43} = \frac{1}{2}\left(\frac{\Gamma_{\rm ej}}{\Gamma} + \frac{\Gamma}{\Gamma_{\rm ej}}\right), \qquad \beta_{43} = \frac{\Gamma_{\rm ej}^2 - \Gamma^2}{\Gamma_{\rm ej}^2 + \Gamma^2}. \tag{4}$$

Thus, given the parameters of regions 1 and 4, only one parameter of the blast wave is left free — Γ. In particular, denoting $\rho_{\rm ej} = \rho_4$, we find

$$\frac{p_r}{p_f} = \frac{\rho_{\rm ej}}{4\Gamma_{\rm ej}^2\rho_1}\left(\frac{\Gamma_{\rm ej}^2}{\Gamma^2} - 1\right)^2. \tag{5}$$

If the pressure balance $p_f = p_r$ is assumed, it immediately determines the instantaneous Γ,

$$\Gamma = \Gamma_{\rm ej}\left[1 + 2\Gamma_{\rm ej}\left(\frac{\rho_1}{\rho_{\rm ej}}\right)^{1/2}\right]^{-1/2}, \qquad p_f = p_r. \tag{6}$$

The problems of this approximation have been mentioned in § 1. In particular, energy conservation is not satisfied and requires a different solution.

3. MECHANICAL MODEL

The gas in the blast wave flows radially with 4-velocity $u^\alpha = (\gamma, \gamma\beta, 0, 0)$ in spherical coordinates (t, r, θ, ϕ). Rest-mass conservation $\nabla_\mu(\rho u^\mu) = r^{-2}\partial_\mu(r^2\rho u^\mu) = 0$ gives

$$\frac{1}{r^2}\frac{d}{dt}\left(r^2\rho\gamma\right) + \rho\gamma\,\frac{\partial\beta}{\partial r} = 0, \tag{7}$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \beta\frac{\partial}{\partial r}$ is the convective derivative (hereafter we use units $c = 1$). Four-momentum conservation $\nabla_\mu T_\alpha^{\ \mu} = 0$ gives two independent equations ($\alpha = 0, 1$)

$$\frac{1}{r^2}\frac{d}{dt}\left(r^2 h\gamma u_\alpha\right) + h\gamma u_\alpha\frac{\partial\beta}{\partial r} + \partial_\alpha p = 0, \tag{8}$$

where $h = e + p$. Instead of $\nabla_\mu T_0^{\ \mu} = 0$ we will use the projection $u^\alpha\nabla_\mu T_\alpha^{\ \mu} = 0$ which yields $r^{-2}\partial_\mu(r^2 h u^\mu) = \gamma dp/dt$. So, conservation of 4-momentum is expressed by

$$\frac{1}{r^2}\frac{d}{dt}\left(r^2 h\gamma^2\beta\right) = -\frac{\partial p}{\partial r} - h\gamma^2\beta\frac{\partial\beta}{\partial r}, \qquad \frac{1}{r^2}\frac{d}{dt}\left(r^2 h\gamma\right) = \gamma\,\frac{dp}{dt} - h\gamma\,\frac{\partial\beta}{\partial r}. \tag{9}$$

We apply equations (7) and (9) to the gas between FS and RS and make the approximation

$$\gamma(t, r) = \Gamma(t), \qquad \partial\beta/\partial r = 0, \qquad r_r < r < r_f, \tag{10}$$

where $r_r(t)$ and $r_f(t)$ are the instantaneous radii of RS and FS, respectively. Then the integration of equations (7) and (9) over r between RS and FS (at $t = const$) yields

$$\begin{aligned}
\frac{\Gamma}{r^2}\frac{d}{dr}\left(r^2\,\Sigma\,\Gamma\right) &= \rho_r(\beta - \beta_r)\Gamma^2 + \frac{1}{4}\,\rho_f, & (11)\\
\frac{1}{r^2}\frac{d}{dr}\left(r^2 H\Gamma^2\right) &= h_r\,(\beta - \beta_r)\Gamma^2 + p_r, & (12)\\
\frac{\Gamma}{r^2}\frac{d}{dr}\left(r^2 H\,\Gamma\right) &= \Gamma^2\frac{dP}{dr} + (h_r - p_r)(\beta - \beta_r)\Gamma^2 + \frac{3}{4}\,p_f, & (13)
\end{aligned}$$

where $\Sigma \equiv \int_{r_r}^{r_f}\rho\,dr$, $H \equiv \int_{r_r}^{r_f} h\,dr$, $P \equiv \int_{r_r}^{r_f} p\,dr$, $\beta_r = dr_r/dt$, and $\beta_f = dr_f/dt$. In the derivation of equations (11)-(13) we made use of $\Gamma \gg 1$: (1) the relativistic blast is a very thin shell, $r_f - r_r \sim r/\Gamma^2 \ll r$, so we used $r_f \approx r_r \approx r$ when calculating the integrals, (2) we used the jump condition at the FS $\beta_f - \beta = 1/4\Gamma^2$ and $h_f = 4p_f \gg \rho_f$, (3) the convective derivative d/dt has been replaced by $\beta d/dr \approx d/dr$ and $\Gamma^2\beta$ by Γ^2 in equation (12). Besides, the last term $\rho_f/4$ in equation (11) may be neglected.

RS may not be relativistic. We derive from the jump conditions for a RS with $\gamma_{43} \ll \Gamma$,

$$\beta - \beta_r = \frac{\Gamma_{\rm ej}^2 - \Gamma^2}{2\Gamma^2(\Gamma_{\rm ej}^2 + 2\Gamma^2)}, \tag{14}$$

$$\rho_r = 2\left(\frac{\Gamma_{\rm ej}}{\Gamma} + \frac{\Gamma}{\Gamma_{\rm ej}}\right)\rho_{\rm ej}, \qquad p_r = \frac{1}{3}\left(\frac{\Gamma_{\rm ej}}{\Gamma} - \frac{\Gamma}{\Gamma_{\rm ej}}\right)^2\rho_{\rm ej}, \qquad h_r = \frac{4}{3}\left(\frac{\Gamma_{\rm ej}^2}{\Gamma^2} + \frac{\Gamma^2}{\Gamma_{\rm ej}^2} + 1\right)\rho_{\rm ej}. \tag{15}$$

This leaves four unknowns in equations (11)-(13): Σ, H, P, and Γ. One more equation is required to close the set of equations. We propose the following approximate relation,

$$H - \Sigma = 4P. \tag{16}$$

As shown below, it is accurate in both limits of ultra-relativistic and non-relativistic RS, and should be a reasonable approximation in an intermediate case.

3.1. Relativistic Reverse Shock

In the case of relativistic RS, $\gamma_{43} \gg 1$, the gas is relativistically hot in both regions 2 and 3, and obeys the equation of state $h = 4p$ everywhere in the blast. Then Σ may be neglected in equation (16) and the condition $H = 4P$ closes the set of mechanical equations. Besides, we have $\beta - \beta_r = (2\Gamma^2)^{-1}$ (since $\Gamma_{\rm ej} \gg \Gamma$, see eq. 14). Equations (11)-(13) then read

$$\frac{\Gamma}{r^2}\frac{d}{dr}\left(r^2\Sigma\Gamma\right) = \frac{1}{2}\rho_r + \frac{1}{4}\rho_f, \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2 H\Gamma^2\right) = 3p_r, \quad \frac{\Gamma}{r^2}\frac{d}{dr}\left(r^2 H\Gamma\right) = \frac{\Gamma^2}{4}\frac{dH}{dr} + \frac{3}{2}p_r + \frac{3}{4}p_f.$$

From the last two equations we find $H(r)$, then substitute to the second equation and get the final differential equation for Γ,

$$\frac{dH}{dr} + \frac{4H}{r} = 4\rho_1, \qquad H(r) = \frac{4}{r^4}\int_0^r \rho_1 r^4 dr. \tag{17}$$

$$\frac{r}{\Gamma}\frac{d\Gamma}{dr} = \frac{r^5}{8}\left(\int_0^r \rho_1 r^4 dr\right)^{-1}\left(\frac{\Gamma_{\rm ej}^2\rho_{\rm ej}}{\Gamma^4} - 4\rho_1\right) + 1. \tag{18}$$

We have used here the jump condition $p_r = \frac{1}{3}(\Gamma_{\rm ej}/\Gamma)^2\rho_{\rm ej}$ for the relativistic RS (see eq. 15). For external medium with a power-law density profile $\rho_1 \propto r^{-k}$, one can simplify $\int \rho_1 r^4 dr = r^5\rho_1/(5-k)$. For example, if $\rho_1 \propto r^{-2}$ (wind-type external medium), $\Gamma_{\rm ej} = const$ and $\rho_{\rm ej} \propto r^{-2}$, then the solution is $\Gamma = const$ and we find

$$p_f = 3p_r, \qquad \Gamma = \left(\frac{3\Gamma_{\rm ej}^2\rho_{\rm ej}}{4\rho_1}\right)^{1/4}, \qquad k = 2. \tag{19}$$

3.2. Non-relativistic Reverse Shock

In the case of non-relativistic RS, $\beta_{43} \ll 1$, we have

$$\rho_r = 4\rho_{\rm ej}, \qquad p_r = \frac{4}{3}\beta_{43}^2\rho_{\rm ej}, \qquad \beta_{43} = 1 - \frac{\Gamma}{\Gamma_{\rm ej}}, \qquad \beta - \beta_r = \frac{\beta_{43}}{3\Gamma_{\rm ej}^2},$$

and the equation of state $u_r = \frac{3}{2}p_r$ and $h_r = \rho_r + u_r + p_r \approx \rho_r$. The effective inertial mass of the blast H is dominated by region 3 as long as $p_r/p_f \gg \beta_{43}$. Indeed, the thicknesses of regions 2 and 3 are $\sim r/\Gamma^2$ and $\sim \beta_{43}r/\Gamma^2$, respectively, and the contributions to H from regions 2 and 3 are $H_2 \sim h_f r/\Gamma^2$ and $H_3 \sim \rho_r\beta_{43}r/\Gamma^2$. This gives $H_2/H_3 \sim (p_f/p_r)\beta_{43} \ll 1$. On the other hand, $H - \Sigma$ and integrated pressure P of the blast is dominated by region 2. In particular, $P_2/P_3 \sim (p_f/p_r)\beta_{43}^{-1} \gg 1$. Therefore the relation $H - \Sigma = 4P$ remains correct for blast waves with non-relativistic RS. Using this relation and subtracting equation (11) from (13), we derive the equation for P in the leading order of β_{43},

$$\frac{4}{r^2\Gamma_{\rm ej}}\frac{d}{dr}\left(r^2 P\Gamma_{\rm ej}\right) = \frac{dP}{dr} + \rho_1. \tag{20}$$

Parameters of the RS do not enter this equation. The ejecta plays the role of a piston that sweeps up the external medium with $\Gamma \approx \Gamma_{\rm ej}$, and equation (20) in essence describes the shocked external medium ahead of the piston. In the case of $\Gamma_{\rm ej} = const$, equation (20) has the solution,

$$P(r) = \frac{1}{3}\,r^{-8/3}\int_0^r r^{8/3}\rho_1(r)\,dr. \tag{21}$$

Subtracting equation (13) from (12), we find $H\Gamma\frac{d\Gamma}{dr} = -\Gamma^2\frac{dP}{dr} + p_r(\beta-\beta_r)\Gamma^2 + p_r - \frac{3}{4}p_f$, which gives, in the leading order of β_{43},

$$\Sigma\frac{d\beta_{43}}{dr} = -\frac{8}{3}\,\frac{P}{r} + \frac{4}{3}\,\rho_1 - \frac{p_r}{\Gamma_{\rm ej}^2}. \tag{22}$$

Finally, equation (11) in the leading order becomes

$$\frac{1}{r^2}\,\frac{d}{dr}\left(r^2\Sigma\right) = \frac{4}{3}\,\rho_{\rm ej}\frac{\beta_{43}}{\Gamma_{\rm ej}^2}. \tag{23}$$

From equations (22) and (23) we find

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\Sigma\,\beta_{43}\right) = -\frac{8}{3}\,\frac{P}{r} + \frac{4}{3}\,\rho_1. \tag{24}$$

Analytical solutions can be found for β_{43} and Σ when the external medium has a power-law density profile, $\rho_1 \propto r^{-k}$, $\Gamma_{\rm ej} = const$, and $\rho_{\rm ej} \propto r^{-2}$. Then $P(r) = \rho_1 r/(11-3k)$,

$$\Sigma^2 = \frac{32}{3}\,\frac{\rho_1\rho_{\rm ej}r^2}{(11-3k)(4-k)\Gamma_{\rm ej}^2}, \qquad \beta_{43}^2 = \frac{3}{2}\frac{(4-k)\Gamma_{\rm ej}^2\rho_1}{(11-3k)\rho_{\rm ej}}, \qquad \frac{p_f}{p_r} = \frac{2}{3}\left(\frac{11-3k}{4-k}\right). \tag{25}$$

4. DISCUSSION

The RS may change from non-relativistic to relativistic as the blast wave propagates. The mechanical model formulated in equations (11)-(16) is applicable to such a general case. The solution can be found numerically for any given external medium $\rho_1(r)$ and ejecta $\Gamma_{\rm ej}(t,r)$ and $\rho_{\rm ej}(t,r)$. Explosions in media with inhomogeneities can be studied, as well as "refreshed" explosions with an energetic tail of the ejecta. Examples of such blast waves and their afterglow emission will be presented elsewhere (Uhm & Beloborodov, in preparation).

The RS shock becomes unimportant for the blast-wave dynamics when $p_r \ll p_f$. This transition is consistently described by the mechanical model. The time of the transition depends on the structure of ejecta. Recent *Swift* observations of the plateau in the X-ray light curve at $t_{\rm obs} < t_{\rm plateau} \sim 10^3 - 10^4$ s suggest that an energetic tail of the ejecta keeps pushing the blast as long as 10^4 s in observer's time $t_{\rm obs} \sim r/2c\Gamma^2$ (Nousek et al. 2006).

The solution at late stages with $p_r \ll p_f$ is obtained from dynamical equations (12)-(13) by setting $p_r = 0$. In particular, for external medium with $\rho_1 \propto r^{-k}$, we find

$$\Gamma^2 = \frac{(5-k)E}{16\pi\rho_1 r^3}, \qquad p_r \ll p_f, \tag{26}$$

where E is the isotropic energy of explosion. This is close to the exact self-similar solution of the full hydrodynamical equations (Blandford & McKee 1976) except the numerical factor: Γ of the mechanical model is lower than the exact Lorentz factor behind FS by a factor of $[(5-k)/(17-4k)]^{1/2}$. This difference is caused by the steep profile of the gas Lorentz factor $\gamma(t,r)$ behind the FS at late stages. Then the approximation of mechanical model ($\partial\gamma/\partial r = 0$ between RS and FS) becomes less accurate, however, still gives a reasonable solution. At later stages, the gas motion becomes non-radial and the blast-wave equations must include the θ-dependence (e.g. Kumar & Granot 2003).

The mechanical model admits a similar formulation for the case of strongly magnetized ejecta (Zhang & Kobayashi 2005 and refs. therein) if the jump conditions are changed accordingly. It can be extended to include the neutron component of the ejecta (Derishev, Kocharovsky, & Kocharovsky 1999; Beloborodov 2003 and refs. therein) and the effects of $e^{\pm}$-loading and preacceleration of external medium by the prompt γ-rays (Thompson & Madau 2000; Beloborodov 2002).

This work was supported by NASA grant NAG5-13382 and Swift grant.

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